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Asymptotics of integrals

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Here some standard methods in asymptotic expansions [1] of integrals are illustrated. These methods are known as *Watson's lemma* and *Laplace's method*. Watson's lemma dates from at latest [Watson 1918a], and Laplace's method at latest from [Laplace 1774]. The exposition here is revisionist, in the sense that we prove a version of Laplace's method by reducing it to Watson's lemma.

For example, we obtain a robust argument for Stirling's formula for $\Gamma(s)$

$$\Gamma(s) \sim \sqrt{2\pi}\, e^{-s}\, s^{s-\frac{1}{2}} \qquad (\text{as } |s| \longrightarrow \infty, \text{ with } \mathrm{Re}\,(s) \geq \delta > 0)$$

and obtain a useful result about ratios of gamma functions,

$$\frac{\Gamma(s+a)}{\Gamma(s)} \sim s^a \qquad (\text{as } |s| \longrightarrow \infty, \text{ for fixed } a, \text{ for } \mathrm{Re}\,(s) \geq \delta > 0)$$

The latter is awkward to obtain as a corollary from Stirling's formula. Laplace's method is further illustrated by an application to asymptotics of functions closely related to Bessel functions, namely,

$$\sqrt{y} \int_0^\infty e^{-(u+\frac{1}{u})y}\, u^{i\nu}\, \frac{du}{u} \sim \sqrt{\pi} \cdot e^{-2y} \qquad (\text{as } y \longrightarrow +\infty)$$

One point is avoidance of standard but immemorable arguments special to the gamma function. Of course, these special arguments do bear more forcefully upon gamma itself: see [Whittaker-Watson 1927] or [Lebedev 1963]. However, to the extent possible, we want to understand the asymptotics of gamma and other important special functions on general principles.

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[1] The simplest notion of *asymptotic* $F(s)$ for $f(s)$ as s goes to $+\infty$ on $\mathbb{R}$, or in a sector in $\mathbb{C}$, is a simpler function $F(s)$ such that $\lim_s f(s)/F(s) = 1$, written $f \sim F$. One might require an error estimate, for example,

$$f \sim F \iff f(s) = F(s) \cdot (1 + O\Big(\frac{1}{|s|}\Big))$$

A more precise form is to say that

$$f(s) \sim f_0(s) \cdot \Big(\frac{c_0}{s^\alpha} + \frac{c_1}{s^{\alpha+1}} + \frac{c_2}{s^{\alpha+2}} + \ldots\Big)$$

(with any auxiliary function f_0) is an *asymptotic expansion* for f when

$$f = f_0(s) \cdot \Big(\frac{c_0}{s^\alpha} + \frac{c_1}{s^{\alpha+1}} + \ldots + \frac{c_n}{s^{\alpha+n}} + O\Big(\frac{1}{|s|^{\alpha+n+1}}\Big)\Big)$$

1. *Heuristic for Stirling's asymptotic*

First we give a heuristic and mnemonic for the main term of Stirling's formula, namely

$$\Gamma(s) \;\sim\; e^{-s} \cdot s^{s-\frac{1}{2}} \cdot \sqrt{2\pi}$$

Using Euler's integral,

$$s \cdot \Gamma(s) \;=\; \Gamma(s+1) \;=\; \int_0^\infty e^{-u}\, u^{s+1}\, \frac{du}{u} \;=\; \int_0^\infty e^{-u}\, u^s\, du \;=\; \int_0^\infty e^{-u+s\log u}\, du$$

The trick is to replace the exponent $-u+s\log u$ by the quadratic polynomial in u best approximating it near its maximum, and evaluate the resulting integral. This replacement can be justified via Watson's lemma and Laplace's method, below, but the heuristic is simpler than the justification.

More precisely, the exponent takes its maximum where its derivative vanishes, at the unique solution $u_o = s$ of

$$-1 + \frac{s}{u} \;=\; 0$$

The second derivative in u of the exponent is $-s/u^2$, which takes value $-1/s$ at $u_o = s$. Thus, near $u_o = s$, the quadratic Taylor-Maclaurin polynomial in t approximating the exponent is

$$-s + s\log s - \frac{1}{2!\, s} \cdot (u-s)^2$$

Thus, we imagine that

$$s \cdot \Gamma(s) \;\sim\; \int_0^\infty e^{-s+s\log s - \frac{1}{2s}\cdot(u-s)^2}\, du \;=\; e^{-s} \cdot s^s \cdot \int_{-\infty}^\infty e^{-\frac{1}{2s}\cdot(u-s)^2}\, du$$

Note that the latter integral is taken over the whole real line. [2] To simplify the remaining integral, replace u by su and cancel a factor of s from both sides,

$$\Gamma(s) \;\sim\; e^{-s} \cdot s^s \cdot \int_{-\infty}^\infty e^{-s(u-1)^2/2}\, du$$

Then replace u by $u+1$, and then u by $u \cdot \sqrt{2\pi/s}$, obtaining

$$\int_{-\infty}^\infty e^{-s(u-1)^2/2}\, du \;=\; \int_{-\infty}^\infty e^{-su^2/2}\, du \;=\; \frac{\sqrt{2\pi}}{\sqrt{s}} \int_{-\infty}^\infty e^{-\pi u^2}\, du \;=\; \frac{\sqrt{2\pi}}{\sqrt{s}}$$

We obtain

$$\Gamma(s) \;\sim\; e^{-s} \cdot s^{s-\frac{1}{2}} \cdot \sqrt{2\pi}$$

Several aspects of this heuristic are dubious, so it is striking that it can be made rigorous, as below.

[2] Evaluation of the integral over the whole line, and simple estimates on the integral over $(-\infty, 0]$, show that the integral over $(-\infty, 0]$ is of a lower order of magnitude than the whole. Thus, the leading term of the asymptotics of the integral over the whole line is the same than the integral from 0 to $+\infty$.

2. *Watson's lemma*

The often-rediscovered *Watson's lemma*[3] gives an asymptotic expansion for certain Laplace transforms, valid in half-planes in $\mathbb{C}$. For example, let h be a smooth function on $(0, +\infty)$ all whose derivatives are of polynomial growth, and expressible for small $x > 0$ as

$$h(x) \;=\; x^\alpha \cdot g(x)$$

for some $\alpha \in \mathbb{C}$, where $g(x)$ is differentiable[4] on $\mathbb{R}$ near 0. Thus, $h(x)$ has an expression

$$h(x) \;=\; x^\alpha \cdot \sum_{n=0}^{\infty} c_n\, x^n \qquad\qquad \text{(for } 0 < x \text{ sufficiently small)}$$

Then there is an *asymptotic expansion* of the Laplace transform of h,

$$\int_0^\infty e^{-xs}\, h(x)\, \frac{dx}{x} \;\sim\; \frac{\Gamma(\alpha)\, c_0}{s^\alpha} + \frac{\Gamma(\alpha+1)\, c_1}{s^{\alpha+1}} + \frac{\Gamma(\alpha+2)\, c_2}{s^{\alpha+2}} + \ldots \qquad\qquad \text{(for } \mathrm{Re}\,(s) > 0)$$

A simple corollary of the error estimates given below is that, letting $\mathrm{Re}\,(\alpha) + 1 - \varepsilon$ be the greatest integer less than or equal $\mathrm{Re}\,(\alpha) + 1$,

$$\int_0^\infty e^{-xs}\, h(x)\, \frac{dx}{x} \;=\; \int_0^\infty e^{-xs}\, x^\alpha\, g(x)\, \frac{dx}{x} \;=\; \frac{\Gamma(\alpha)\, g(0)}{s^\alpha} + O\Big(\frac{1}{|s|^{\mathrm{Re}\,(\alpha)+1-\varepsilon}}\Big)$$

Since

$$\mathrm{Re}\,(\alpha) + 1 - \varepsilon \;>\; \mathrm{Re}\,(\alpha)$$

the error term is of strictly smaller order of magnitude in s.

The idea of the proof is straightforward: the expansion is obtained from

$$\int_0^\infty e^{-xs}\, h(x)\, \frac{dx}{x} \;=\; \int_0^\infty e^{-xs}\, x^\alpha \big(c_0 + \ldots + c_n x^n\big)\, \frac{dx}{x} \;+\; \int_0^\infty e^{-xs}\, x^\alpha \Big(g(x) \;-\; \big(c_0 + \ldots + c_n x^n\big)\Big)\, \frac{dx}{x}$$

The first integral gives the asymptotic expansion, and for $\mathrm{Re}\,(s) > 0$ the second integral can be integrated by parts essentially $\mathrm{Re}\,(\alpha) + n$ times and trivially bounded to give a $O(1/s^{\alpha+n-\varepsilon})$ error term for some small $\varepsilon \geq 0$. Note that for the integration by parts the denominator x in the measure must be moved into the integrand proper, accounting for a slight reduction of the order of vanishing of the integrand at 0.

To understand the error, let $\varepsilon \geq 0$ be the smallest such that

$$N \;=\; \mathrm{Re}\,(\alpha) + n - \varepsilon \;\in\; \mathbb{Z}$$

The subtraction of the initial polynomial and re-allocation of the $1/x$ from the measure makes $x^{\alpha-1}(g(x) - (c_0 + \ldots + c_n x^n)$ vanish to order N at 0. This, with the exponential e^{-sx} and the presumed

[3] This lemma appeared in the treatise [Watson 1922] on page 236, citing [Watson 1918a], page 133. Curiously, the aggregate bibliography of [Watson 1922] omitted [Watson 1918a], and the footnote mentioning it gave no title. Happily, [Watson 1918a] is mentioned by title in [Blaustein-Handelsman 1975]. In the bibliography at the end, we mention [Watson 1917], [Watson 1918a], [Watson 1918b], for perspective.

[4] We do *not* need to assume that g is *real-analytic* near 0, only that it and its derivatives have finite Taylor expansions approximating it well as $x \to 0^+$.

polynomial growth of h and its derivatives, allows integration by parts N times without boundary terms, giving

$$\int_0^\infty e^{-xs}\,h(x)\,dx \;=\; \frac{\Gamma(\alpha)\,c_0}{s^\alpha} + \frac{\Gamma(\alpha+1)\,c_1}{s^{\alpha+1}} + \ldots + \frac{\Gamma(\alpha+n)\,c_n}{s^{\alpha+n}}$$
$$+\frac{1}{s^N}\int_0^\infty e^{-sx}\left(\frac{\partial}{\partial x}\right)^N \Big(x^\alpha\cdot\big(g(x)-(c_0+\ldots+c_n x^n)\big)\Big)\,dx$$

The last error-like term is $O(s^{-[\mathrm{Re}\,(\alpha)+n-\varepsilon]})$. That is, computing in this fashion, the error term swallows up the last term in the asymptotic expansion.

Visibly, this argument applies to more general sorts of expansions near 0.

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3. *Watson's lemma illustrated on* $B(s,a)$

Here is an important example of an asymptotic result non-trivial to derive from Stirling's formula for $\Gamma(s)$, but easy to obtain from Watson's lemma. Euler's beta integral is[5]

$$B(s,a) \;=\; \int_0^1 x^{s-1}\,(1-x)^{a-1}\,dx \;=\; \frac{\Gamma(s)\,\Gamma(a)}{\Gamma(s+a)}$$

Fix a with $\mathrm{Re}\,(a)>0$, and consider this integral as a function of s. Letting $x=e^{-u}$ gives an integrand fitting Watson's lemma,

$$B(s,a) \;=\; \int_0^\infty e^{-su}\,(1-e^{-u})^{a-1}\,du \;=\; \int_0^\infty e^{-su}\,(u-\frac{u^2}{2!}+\ldots)^{a-1}\,du$$
$$=\; \int_0^\infty e^{-su}\,u^a\cdot(1-\frac{u}{2!}+\ldots)^{a-1}\,\frac{du}{u} \;\sim\; \frac{\Gamma(a)}{s^a}$$

taking just the first term in an asymptotic expansion, using Watson's lemma. Thus,

$$\frac{\Gamma(s)\,\Gamma(a)}{\Gamma(s+a)} \;\sim\; \frac{\Gamma(a)}{s^a}$$

giving

$$\frac{\Gamma(s)}{\Gamma(s+a)} \;\sim\; \frac{1}{s^a} \qquad\qquad \text{(for } a \text{ fixed)}$$

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4. *Simple form of Laplace's method*

A simple version of Laplace's method[6] obtains asymptotics in s for certain integrals of the form

$$\int_0^\infty e^{-s\cdot f(u)}\,du$$

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[5] We recall how to obtain the expression for beta in terms of gamma. With $x=u/(u+1)$ in the beta integral,

$$B(s,a) \;=\; \int_0^\infty u^{s-1}\,(u+1)^{-(s-1)-(a-1)-2}\,du \;=\; \int_0^\infty u^{s-1}\,(u+1)^{-s-a}\,du$$
$$=\; \frac{1}{\Gamma(s+a)}\int_0^\infty\!\int_0^\infty u^s\,e^{-v(u+1)}\,v^{s+a}\,\frac{dv}{v}\,\frac{du}{u}$$

using $\int_0^\infty e^{-vy}\,v^b\,dv/v = \Gamma(b)/y^b$. Replacing u by u/v gives $B(s,a)=\Gamma(s)\Gamma(a)/\Gamma(s+a)$.

[6] Perhaps the first appearance of this is in [Laplace 1774].

with f real-valued. The idea is that the *minimum values* of $f(u)$ should dominate, and the leading term of the asymptotics should be

$$\int_0^\infty e^{-s\cdot f(u)}\,du \;\sim\; e^{-sf(u_o)}\cdot\frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}} \qquad\text{(for } |s|\longrightarrow\infty\text{, with } \mathrm{Re}(s)\geq\delta>0)$$

To reduce this to Watson's lemma, break the integral at points where the derivative f' changes sign, and change variables to convert each fragment to a Watson-lemma integral. For Watson's lemma to be legitimately applied, we will find that f must be smooth with all derivatives of at most polynomial growth *and* at most polynomial *decay*, as $u\longrightarrow+\infty$.

For simplicity *assume* that there is exactly *one* point u_o at which $f'(u_o)=0$, and that $f''(u_o)>0$. Further, assume that $f(u)$ goes to $+\infty$ at 0^+ and at $+\infty$. Since $f'(u)>0$ for $u>u_o$ and $f'(u)<0$ for $0<u<u_o$, on each of these two intervals there is a smooth square root $\sqrt{f(u)-f(u_o)}$ and there are smooth functions F,G such that

$$\begin{cases} F(\sqrt{f(u)-f(u_o)}) &=& u \qquad\text{(for } u_o<u<+\infty)\\ G(\sqrt{f(u)-f(u_o)}) &=& u \qquad\text{(for } 0<u<u_o)\end{cases}$$

Then

$$\int_0^\infty e^{-s\,f(u)}\,du \;=\; e^{-sf(u_o)}\int_0^{u_o} e^{-s\,(f(u)-f(u_o))}\,du + e^{-sf(u_o)}\int_{u_o}^\infty e^{-s\,(f(u)-f(u_o))}\,du$$

$$= e^{-sf(u_o)}\Big(\int_0^\infty e^{-s\,x^2}\,F'(x)\,dx + \int_0^\infty e^{-s\,x^2}\,G'(x)\,dx\Big)$$

by letting $x=\sqrt{f(u)-f(u_o)}$ in the two intervals. In both integrals, replacing x by $\sqrt{x}$ gives Watson's-lemma integrals

$$\int_0^\infty e^{-s\,f(u)}\,du \;=\; e^{-sf(u_o)}\Big(\int_0^\infty e^{-sx}\,\tfrac{1}{2}x^{1/2}\,F'(\sqrt{x})\,\frac{dx}{x} + \int_0^\infty e^{-sx}\,\tfrac{1}{2}x^{1/2}\,G'(\sqrt{x})\,\frac{dx}{x}\Big)$$

At this point the needed conditions on F, hence, on f, become clear: since F must be smooth with all derivatives of at most polynomial growth, direct chain-rule computations show that it suffices that no derivative of f increases *or decreases* faster than polynomially as $u\longrightarrow+\infty$. The assumptions $f'(u_o)=0$ and $f''(u_o)>0$ assure that F has a Taylor series expansion near 0, giving a suitable expansion

$$\tfrac{1}{2}x^{1/2}F'(x) \;=\; \tfrac{1}{2}F'(0)x^{1/2} + \frac{\tfrac{1}{2}F^{(2)}(0)}{1!}x^{3/2} + \frac{\tfrac{1}{2}F^{(3)}(0)}{2!}x^{5/2} + \frac{\tfrac{1}{2}F^{(4)}(0)}{3!}x^{7/2} + \ldots \qquad\text{(small } x>0)$$

From this, the main term of the Watson's lemma asymptotics for the integral involving F would be

$$\int_0^\infty e^{-sx}\,\tfrac{1}{2}x^{1/2}\,F'(\sqrt{x})\,\frac{dx}{x} \;\sim\; \frac{\Gamma(\frac{1}{2})\,F'(0)}{2}\cdot\frac{1}{\sqrt{s}}$$

To determine $F'(0)$, or any higher coefficients, from $F(x)=u$, we have $F'(x)\cdot\frac{dx}{du}=1$. Since

$$x \;=\; \sqrt{f(u)-f(u_o)} \;=\; \sqrt{(u-u_o)^2\cdot\frac{f''(u_o)}{2!}+\ldots} \;=\; \sqrt{\frac{f''(u_o)}{2}}\cdot\Big((u-u_o)+\ldots\Big)$$

the derivative is

$$\frac{dx}{du} \;=\; \sqrt{\frac{f''(u_o)}{2}}\cdot\Big(1+O(u-u_o)\Big)$$

Thus,

$$F'(x) \;=\; \frac{1}{\frac{dx}{du}} \;=\; \sqrt{\frac{2}{f''(u_o)}}\cdot\Big(1+O(u-u_o)\Big)$$